A Note on Path Congruence in Non-Deterministic Systems

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April 21, 2026

Abstract

We consider non-deterministic systems in which multiple transformation paths may connect the same pair of states. We isolate a structural configuration in which an equivalence relation on such paths is generated internally by the system and is required to be stable under composition. The resulting structure may be viewed as a path category equipped with a congruence on morphisms, admitting a natural interpretation in terms of dynamic invariance. The note is descriptive and does not attempt a general theory.

1. Introduction

In a variety of settings, one encounters systems in which a given initial state may evolve along multiple distinct sequences of transformations. Such behavior arises in rewriting systems, transition systems, and related constructions, where the same endpoints may be connected by different transformation paths. In these contexts, it is natural to ask when two such paths should be regarded as equivalent.

Standard approaches treat this question in different ways. In rewriting theory, one studies relations between derivations through notions such as confluence or joinability. In categorical settings, equivalence between parallel morphisms may be imposed via congruences or expressed through higher morphisms. In concurrency theory, one encounters behavioral equivalences defined in terms of indistinguishability under observation. In each case, however, the equivalence relation is typically either derived from an external criterion or specified as additional structure.

The purpose of this note is to describe a configuration in which an equivalence relation on transformation paths is taken to be intrinsic to the system and is required to satisfy a strong form of compatibility with further evolution. The emphasis is on isolating the structural requirements rather than on developing a general theory.

2. Path Structure

Let S be a set of states and T a collection of transformations. We assume that the system is non-deterministic in the sense that, for a given state and transformation, multiple successor states may be possible. Concretely, one may regard the system as specified by a directed multigraph whose vertices are elements of S and whose edges are labeled by elements of T, with multiple edges allowed between the same pair of states.

A **path** from a state x to a state y is a finite sequence of composable transformations beginning at x and ending at y. We write $q \circ p$ for the composition of paths, meaning first p then q, whenever the target of p matches the source of q. Each state admits an empty path serving as an identity.

In this way one obtains the path category $\mathsf{Path}(S, T)$, whose objects are states and whose morphisms are paths.

We assume that there exist distinct paths $p \neq q$ with the same source and target. No assumption of confluence is made; in particular, distinct paths may lead to genuinely different intermediate or terminal configurations.

3. Path Congruence

We suppose that, for each pair of states $x, y \in S$, an equivalence relation $\sim$ is given on the set of paths from x to y. We require that $\sim$ be compatible with composition in the following sense:

$$p \sim q \implies r \circ p \circ s \sim r \circ q \circ s$$

whenever the compositions are defined. Thus $\sim$ is a congruence on the category $\mathsf{Path}(S, T)$.

We further assume that $\sim$ is nontrivial, in the sense that there exist distinct paths $p \neq q$ with $p \sim q$.

The equivalence relation $\sim$ is not taken to be freely specified. Rather, it is understood as arising from the internal structure of the system. One may formalize this by specifying a family of local identifications between short paths and taking $\sim$ to be the smallest equivalence relation containing these identifications and closed under composition. In other words, $\sim$ is the congruence generated by a given set of elementary identifications internal to the system.

4. Dynamic Interpretation

The congruence condition may be viewed as expressing a form of invariance under extension. If $p \sim q$ are paths from x to y, then for any path r beginning at y, the composed paths $r \circ p$ and $r \circ q$ remain equivalent.

In this sense, once two paths have been identified, no further sequence of transformations can distinguish between them. The equivalence relation therefore captures not only a relationship between past histories, but also a constraint on how those histories may be used as inputs to future transformations.

This interpretation does not imply confluence: distinct paths may remain distinct as morphisms in $\mathsf{Path}(S, T)$, and branching behavior is preserved. The condition only asserts that, within a given equivalence class, paths behave identically with respect to all possible extensions.

5. Discussion

The configuration described above may be viewed from several perspectives. From the standpoint of rewriting systems, it corresponds to equipping the space of derivations with a congruence that is stable under extension, rather than focusing solely on equivalence of terms. From a categorical point of view, one is considering a quotient category $\mathsf{Path}(S,T)/{\sim}$, obtained by identifying equivalent paths under a congruence on morphisms. In settings related to concurrency, the condition of invariance under extension resembles behavioral equivalence, though here it is formulated directly at the level of paths.

The purpose of isolating these conditions is not to claim a new framework, but to make explicit a combination of features that may arise when one considers non-deterministic systems together with internally generated identifications of transformation histories.

References

[1] S. Mac Lane, *Categories for the Working Mathematician*, 2nd ed., Springer, 1998.